



EE 232 Lightwave Devices Lecture 5: Time-Dependent Perturbation Theory, Fermi's Golden Rule

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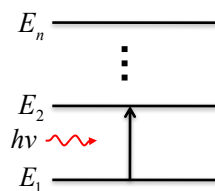
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Time-Dependent Perturbation

Consider a quantum mechanical system:



$$H_0 \phi_n(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} \phi_n(\vec{r}, t)$$

$$\phi_n(\vec{r}, t) = \phi_n(\vec{r}) e^{-\frac{iE_n t}{\hbar}}$$

$\phi_n(\vec{r}) = |n\rangle$ an orthonormal set of eigenstates

$$\langle m | n \rangle = \int \phi_m^*(\vec{r}) \phi_n(\vec{r}) d\vec{r} = \delta_{mn}$$

Consider a single-frequency, time-varying stimulus

$$H'(\vec{r}, t) = H'(\vec{r})e^{-i\omega t} + H'^{\dagger}(\vec{r})e^{i\omega t} \quad \text{for } t > 0$$

$$H = H_0 + H'(\vec{r}, t)$$

$$H\psi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t)$$

Assuming $|H'| \ll |H_0|$

The new wavefunction can be expressed as a linear combination of original eigenstates with time-varying coefficients:

$$\psi(\vec{r}, t) = \sum_n a_n(t) \phi_n(\vec{r}) e^{-iE_n t/\hbar}$$

$|a_n(t)|^2$: probability of electron at state $|n\rangle$ at time t

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Time-Dependent Perturbation (cont'd)

$$H\psi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t)$$

$$(\cancel{H_0} + H') \sum_n a_n(t) \phi_n(\vec{r}) e^{-iE_n t/\hbar} = i\hbar \sum_n \frac{da_n(t)}{dt} \phi_n(t) e^{-iE_n t/\hbar} + i\hbar \sum_n a_n(t) \phi_n(\vec{r}) \left(\frac{-iE}{\hbar} \right) e^{-iE_n t/\hbar}$$

$$H' \sum_n a_n(t) |n\rangle e^{-iE_n t/\hbar} = i\hbar \sum_n \frac{da_n(t)}{dt} |n\rangle e^{-iE_n t/\hbar}$$

Multiply both sides by $\langle m |$ (i.e., multiply by $\phi_m^*(\vec{r})$ and integrate over \vec{r})

$$\sum_n a_n(t) \langle m | H' | n \rangle e^{-iE_n t/\hbar} = i\hbar \sum_n \frac{da_n(t)}{dt} \langle m | n \rangle e^{-iE_n t/\hbar} = i\hbar \frac{da_m(t)}{dt} e^{-iE_m t/\hbar}$$

$$\frac{da_m(t)}{dt} = \frac{1}{i\hbar} \sum_n a_n(t) H'_{mn}(t) e^{i\omega_{mn} t}$$

$$\omega_{mn} = \frac{E_m - E_n}{\hbar}$$

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First-Order Perturbation

To track the order of perturbation, let

$$H = H_0 + \lambda H'$$

$$a_n(t) = a_n^{(0)}(t) + \lambda a_n^{(1)}(t) + \lambda^2 a_n^{(2)}(t) + \dots$$

Group terms with the same order of λ :

$$\frac{da_m^{(0)}(t)}{dt} = 0 \Rightarrow a_m^{(0)}(t) = \text{constant}$$

$$\frac{da_m^{(1)}(t)}{dt} = \frac{1}{i\hbar} \sum_n a_n^{(0)}(t) H'_{mn}(t) e^{i\omega_{mn} t}$$

$$\frac{da_m^{(2)}(t)}{dt} = \frac{1}{i\hbar} \sum_n a_n^{(1)}(t) H'_{mn}(t) e^{i\omega_{mn} t}$$

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First-Order Perturbation (Cont'd)

Initial state i at $t=0$ and final state f

$$\begin{cases} a_i^{(0)}(t) = 1 \\ a_m^{(0)}(t) = 0 \text{ if } m \neq i \end{cases}$$

$$\begin{aligned} \frac{da_f^{(1)}(t)}{dt} &= \frac{1}{i\hbar} H'_{fi}(t) e^{i\omega_f t} = \frac{1}{i\hbar} (H'_{fi} e^{-i\omega t} + H'_{fi}^\dagger e^{i\omega t}) e^{i\omega_f t} \\ &= \frac{1}{i\hbar} (H'_{fi} e^{i(\omega_f - \omega)t} + H'_{fi}^\dagger e^{i(\omega_f + \omega)t}) \end{aligned}$$

$$a_f^{(1)}(t) = \frac{-1}{\hbar} \left(H'_{fi} \frac{e^{i(\omega_f - \omega)t} - 1}{\omega_f - \omega} + H'_{fi}^\dagger \frac{e^{i(\omega_f + \omega)t} - 1}{\omega_f + \omega} \right)$$

We are only interested at frequencies near resonance:

$$|a_f^{(1)}(t)|^2 = \frac{4|H'_{fi}|^2}{\hbar^2} \frac{\sin^2\left(\frac{\omega_f - \omega}{2}t\right)}{(\omega_f - \omega)^2} + \frac{4|H'_{fi}^\dagger|^2}{\hbar^2} \frac{\sin^2\left(\frac{\omega_f + \omega}{2}t\right)}{(\omega_f + \omega)^2}$$

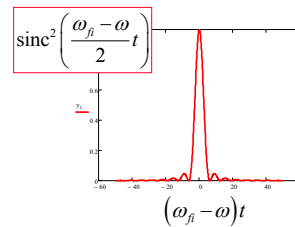
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Fermi's Golden Rule

$$\begin{aligned} \frac{\sin^2\left(\frac{\omega_f - \omega}{2}t\right)}{(\omega_f - \omega)^2} &= \frac{t^2}{4} \text{sinc}^2\left(\frac{\omega_f - \omega}{2}t\right) \\ &\rightarrow \frac{\pi t}{2} \delta(\omega_f - \omega) \text{ as } t \rightarrow \infty \end{aligned}$$



$$|a_f^{(1)}(t)|^2 = \frac{2\pi t |H'_{fi}|^2}{\hbar^2} \delta(\omega_f - \omega) + \frac{2\pi t |H'_{fi}^\dagger|^2}{\hbar^2} \delta(\omega_f + \omega)$$

Transition Rate:

$$W_{i \rightarrow f} = \frac{d}{dt} |a_f^{(1)}(t)|^2 = \frac{2\pi |H'_{fi}|^2}{\hbar^2} \delta(\omega_f - \omega) + \frac{2\pi |H'_{fi}^\dagger|^2}{\hbar^2} \delta(\omega_f + \omega)$$

Note: $\delta(E_f - E_i - \hbar\omega) = \frac{1}{\hbar} \delta(\omega_f - \omega_i - \omega)$

$$W_{i \rightarrow f} = \frac{2\pi |H'_{fi}|^2}{\hbar} \delta(E_f - E_i - \hbar\omega) + \frac{2\pi |H'_{fi}^\dagger|^2}{\hbar} \delta(E_f - E_i + \hbar\omega)$$

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Physical Interpretation

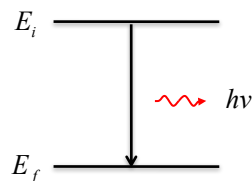
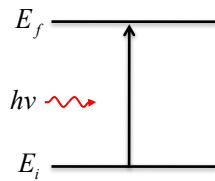
$$W_{i \rightarrow f} = \frac{2\pi |H'_{fi}|^2}{h} \delta(E_f - E_i - h\omega) + \frac{2\pi |H'_{fi}|^2}{h} \delta(E_f - E_i + h\omega)$$

$$E_f = E_i + h\omega$$

Absorption of a photon

$$E_f = E_i - h\omega$$

Emission of a photon



- Conservation of energy
- Transition rate is proportional to the square of the “matrix element”

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Distributed Final States

- If the final state is a distribution of states, the transition rate is proportional to the density of states of the final state:

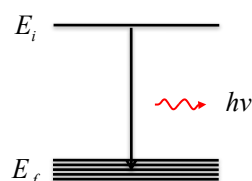
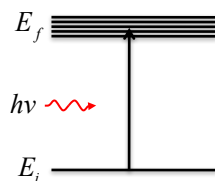
$$W_{i \rightarrow f} = \frac{2\pi |H'_{fi}|^2}{h} \rho_f \delta(E_f - E_i - h\omega) + \frac{2\pi |H'_{fi}|^2}{h} \rho_f \delta(E_f - E_i + h\omega)$$

$$E_f = E_i + h\omega$$

Absorption of a photon

$$E_f = E_i - h\omega$$

Emission of a photon



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